Stationary Convection in Toroidal Plasma Configurations

Horst Wobig

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ABSTRACT

The mathematical structure of the two-fluid MHD equations including plasma-neutral interaction is investigated. We obtain an elliptic system for the plasma density and the electric potential. These equations exhibit a strong similarity to the hydrodynamic equations of thermal convection. The set of equations is applied to a toroidal magnetic field with and without shear and it is shown that magnetic irregularities may cause appreciable convection.

STATIONARY CONVECTION IN TOROIDAL PLASMA CONFIGURATIONS

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I INTRODUCTION

Diffusion in toroidal plasma configurations is one of the major problems in fusion research and has been investigated by both theoreticians and experimentalists for a number of years.

The role of the secondary currents with respect to plasma losses has been discussed in a paper by D. Pfirsch and A. Schlüter (1962), inertia effects have been taken into account by Knorr (1965), and the role of viscosity has been considered by E.T. Karlson (1968). Furthermore, the rotation of the plasma column has been taken into account by T.E. Stringer (1969). For velocities close to the velocity of sound the classical diffusion problem has been investigated by W.K. Winsor et al. (1970). Recently, S. Yoshikawa (1970) has included ion-neutral collisions and calculated the mass flow in terms of spatial density fluctuations. In this model the inertia terms are neglected, and so it only applies to configurations with small mass velocities ($v \ll c_s$). But little attention has been paid to the mathematical structure and the problem of boundary conditions within these models. In the following the mathematical structure of the model used by S. Yoshikawa will be investigated and equations will be applied to the problem of stationary plasma loss in a toroidal system.

II BASIC EQUATIONS

We start from the macroscopic two-fluid equations and neglect the inertia and viscosity terms:

$$0 = -\nabla p_i - en\nabla\phi + en\underline{v}_i \times \underline{B} - \alpha n\underline{v}_i - \beta n(\underline{v}_i - \underline{v}_e)$$
(1)
$$\mathrm{div}\; n\underline{v}_i = Q_i$$

$$0 = -\nabla p_e + en\nabla\phi - en\underline{v}_e \times \underline{B} - \gamma n\underline{v}_e + \beta n(\underline{v}_i - \underline{v}_e)$$
(2)
$$\mathrm{div}\; n\underline{v}_e = Q_e$$

$n = n_e = n_i$ plasma density, $\underline{B}$ vacuum magnetic field (assumption of low ß)

$p_i = nkT_i$, $p_e = nkT_e$, $Q_i = Q_e$ source terms for electrons and ions.

$\alpha = m_i\,\nu_{io}$, $\gamma = m_e\,\nu_{eo}$, $ß = m_e\,\nu_{ei}$, ν_{io}, ν_{eo}, ν_{ei} are collision frequencies.

In these equations we have assumed that the neutrals have no macroscopic motion ($\underline{v}_o = o$). The momentum given to the neutral background is immediately carried to the wall. The plasma is considered to be surrounded by a conducting wall equations (1,2) have to be solved in a bounded domain.

The inertial terms can be neglected if

(3)
$$v_i \ll \frac{L_{\parallel}}{\lambda_{io}}\, v_{th,i} \; ; \quad v_e \ll \frac{L_{\parallel}}{\lambda_{eo}}\, v_{th,e}$$

Here λ_{io}, λ_{eo} are the mean free paths for ion-neutral collisions and electron-neutral collisions, v_{th} is the thermal velocity, and L" is a measure of the plasma inhomogeneity parallel to $\underline{B}$.

The range of validity of eqs. (1, 2) is determined on the following assumptions:

1) The ion-ion collision frequency and electron-electron collisions are high enough to establish a local Maxwellian.

2) The interaction of the neutral particles with the wall is strong, and therefore the distribution function of the neutral particles is Maxwellian.

If these conditions are satisfied, the pressure of ions and electrons is isotropic and the momentum exchange with the neutral background can be described by a term $m_i \ \nu_{io} \ n\underline{v}_i$. This form of the interaction term can be derived from the collision term of the Boltzmann equation. It is also valid in the case of long mean free paths λ_{io}, λ_{eo} and small collision frequencies (ν_{io}, ν_{eo}).

In the following some mathematical details of the system (1), (2) will be investigated and the boundary value problem will be formulated. For simplicity we introduce two-stream functions ψ_i, ψ_e by

(4)
$$\psi_i = -\frac{KT_i}{e} \ln \frac{n}{n_o} - \phi$$
$$\psi_e = -\frac{KT_e}{e} \ln \frac{n}{n_o} + \phi$$
n_o = reference density

Withthese definitions the density can be written:

$$n = n_o \exp\left[-\left(\frac{e\psi_i}{KT_i} + \frac{e\psi_e}{KT_e}\right)\right] \tag{5}$$

We now write eq. (1) in the form

(6)
$$0 = \nabla\psi_i + \underline{v}_i \times \underline{B} - (\alpha + \beta)\underline{v}_i + \beta\underline{v}_e$$
$$0 = \nabla\psi_e - \underline{v}_e \times \underline{B} - (\gamma + \beta)\underline{v}_e + \beta\underline{v}_i$$

The coefficients α, β, γ have been redefined by

$$\alpha = \frac{m_i \nu_{io}}{e} \; ; \; \beta = \frac{m_e \nu_{ei}}{e} \; ; \; \gamma = \frac{m_e \nu_{eo}}{e}$$

With the help of two matrices A, B

$$(7) \qquad A\underline{v} =: \underline{v} \times \underline{B} - (\alpha + \beta)\underline{v} \; ; \; B\underline{v} =: -\underline{v} \times \underline{B} - (\gamma + \beta)\underline{v}$$

the basic equations are simplified to

$$(8) \qquad \begin{aligned} 0 &= \nabla \psi_i + A\underline{v}_i + \beta \underline{v}_e \\ 0 &= \nabla \psi_e + B\underline{v}_e + \beta \underline{v}_i \end{aligned}$$

The matrices A, B are hermitian, negative definite and commutable.

$$(9) \qquad \underline{v} \cdot A\underline{v} = -(\alpha + \beta)\, \underline{v} \cdot \underline{v} \; ; \; \underline{v} \cdot B\underline{v} = -(\gamma + \beta)\, \underline{v} \cdot \underline{v}$$

and $\qquad AB = BA$

Furthermore, the matrix $C := AB - \beta^2$ is positive definite, hermitian and commutable with A and B. This can be verified by straightforward calculations:

$$(10) \qquad \underline{v} \cdot C\underline{v} = (\underline{v} \times \underline{B})^2 + [\alpha\gamma + \beta(\alpha + \gamma)]\, \underline{v} \cdot \underline{v}$$

Therefore C^{-1} exists and the velocities v_i, v_e can be written

$$(11) \qquad \begin{aligned} \underline{v}_i &= C^{-1} [(\beta \nabla \psi_e - B \nabla \psi_i)] \\ \underline{v}_e &= C^{-1} [\beta \nabla \psi_i - A \nabla \psi_e] \end{aligned}$$

As can be seen from (10), this is possible only because $\alpha \neq o$ and $\gamma \neq o$. By introducing (11) into the equations of continuity (2) we obtain a quasilinear system of the second order.

$$\text{div}\; n(\psi_i, \psi_e)\, C^{-1}(\beta \nabla \psi_e - B \nabla \psi_i) = Q_i \tag{12}$$

$$\text{div}\; n(\psi_i, \psi_e)\, C^{-1}(\beta \nabla \psi_i - A \nabla \psi_e) = Q_e$$

In the following we shall show that the system (12) is elliptic.

The general form of system (12) is

$$a_{i,k}^{l,m} \frac{\partial^2 \psi_m}{\partial x_i \partial x_k} + b_k^{l,m} \frac{\partial \psi_e}{\partial x_k} = Q_m \qquad \begin{matrix} i,k = 1,2,3 \\ l,m = e,i \end{matrix} \tag{13}$$

(summation over equal indices)

The ellipticity depends on the matrix

$$\omega^{l,m} = \sum_{i,k} a_{i,k}^{l,m} X_i X_k \tag{14}$$

(X_i is an arbitrary vector)

If det $\{\omega^{l,m}\} \neq o$, the system is called elliptic.

The four elements of $\omega^{l,m}$ are

$$\omega_{1,1} = n\beta\, \underline{X} \cdot C^{-1} \underline{X} \;;\; \omega_{22} = n\beta\, \underline{X} \cdot C^{-1} \underline{X} \tag{15}$$

$$\omega_{12} = -n\, \underline{X} \cdot C^{-1} A \underline{X} \;;\; \omega_{21} = -n\, \underline{X} C^{-1} B \underline{X}$$

The determinant is

$$n^2 \beta (\underline{X} \cdot C^{-1} \underline{X})^2 - n^2 (\underline{X} C^{-1} A \underline{X})(\underline{X} \cdot C^{-1} B \underline{X}) \tag{16}$$

or with $\underline{X} = C\underline{Y}$

$$\det\{\omega^{l,m}\} = n^2\beta(C\underline{Y}\cdot\underline{Y})^2 - n^2(C\underline{Y}\cdot A\underline{Y})(C\underline{Y}\cdot B\underline{Y})$$

A lengthy, but straight-forward, calculation yields

$$(C\underline{Y}\cdot A\underline{Y})(C\underline{Y}\cdot B\underline{Y}) = (\gamma+\beta)(\alpha+\beta)(\underline{Y}\cdot C\underline{Y})^2 \tag{17}$$
$$+(\alpha-\gamma)^2(\underline{Y}\times\underline{B})^2[\alpha\gamma+\beta(\gamma+\alpha)]\,\underline{Y}\cdot\underline{Y}\,.$$

Since the last term in (17) is positive, we find the estimate

$$(C\underline{Y}\cdot A\underline{Y})(C\underline{Y}\cdot B\underline{Y}) \geq (\gamma+\beta)(\alpha+\beta)(\underline{Y}\cdot C\underline{Y})^2$$

and consequently

$$\det\{\omega^{l,m}\} \leq -n^2[\alpha\gamma+\beta(\alpha+\gamma)](\underline{X}\cdot C^{-1}\underline{X})^2$$

Since C is bounded from above and positive, C^{-1} is also positive and bounded from below.

$$\underline{X}\cdot C^{-1}\underline{X} \geq \delta\,\underline{X}\cdot\underline{X}\ ; \qquad \delta > 0$$

δ is the largest eigenvalue of C.

The final result is

$$\det\{\omega^{l,m}\} \leq -n^2[\alpha\gamma+\beta(\alpha+\gamma)]\,\delta\,(\underline{X}\cdot\underline{X})^2 \tag{18}$$

The determinant does not vanish, except in the case $\underline{X} = o$ or $\alpha, \gamma = o$.

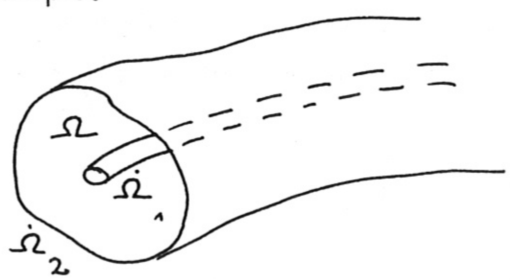
If we neglect the plasma-neutral collisions, the basic equations are no longer elliptic, but this property is not destroyed if we only neglect the electron-neutral collisions or the ion-neutral collisions, but not both. In a weakly ionized plasma electron-ion collisions are unimportant ($\beta = o$), and in this case ($\alpha \neq o, \gamma \neq o$) the system is still elliptic.

As is well known from the theory of elliptic equations, the appropriate boundary conditions are to prescribe the functions ψ_e and ψ_i on the boundary (Dirichlet problem). This is equivalent to giving n and $\emptyset$ on the boundary. The simplest case would be n = const and $\emptyset$ = const on the boundary, but in principle n and $\emptyset$ can be arbitrary continuous functions on $\dot{\Omega}$ ($\dot{\Omega}$ is the boundary of the domain Ω).

In order to avoid singularities in the system (12), we assume that n > o on the boundary.

III. PLASMA LOSS

From eq. (6) it is possible to derive a general formula for the plasma loss. We consider a plasma with a localized plasma source ($Q_{i,e}(x) \neq o$ in a small domain compared with Ω). A stellarator with a source on the magnetic axis would be an example.



ψ_i and ψ_e may be const on $\dot{\Omega}_2$ and $\dot{\Omega}_1$.

From eq. (6) we obtain by partial integration

$$F_i \left(\psi_i^{(1)} - \psi_i^{(2)}\right) = - \int_{\Omega} [n(\alpha+\beta)\underline{v}_i^2 + n\beta\, \underline{v}_e \cdot \underline{v}_e]\, dV \tag{19}$$

$$F_e \left(\psi_e^{(1)} - \psi_e^{(2)}\right) = - \int_{\Omega} [n(\gamma+\beta)\underline{v}_e^2 + n\beta\, \underline{v}_e \cdot \underline{v}_i]\, dV$$

where $F_i = \int_{\dot{\Omega}_2} n\underline{v}_i \cdot \underline{df}$; $F_e = \int_{\dot{\Omega}_2} n\underline{v}_e \cdot \underline{df}$

and $\psi^{(1)}, \psi^{(2)}$ are the boundary values of ψ on $\dot{\Omega}_1, \dot{\Omega}_2$.

From these relations we see that the mass flow F_i, F_e is determined by the boundary conditions on $\dot{\Omega}_1, \dot{\Omega}_2$. In general, there is a mass flow and an electric current from $\dot{\Omega}_1$ to $\dot{\Omega}_2$. If we require vanishing electric current ($F_i = F_e$), the system is overdetermined. In this case ψ_i or ψ_e on $\dot{\Omega}_1$ has to be left "floating". If the mass flow and the electric current are given, the density and the potential on $\dot{\Omega}_1$ are "floating".

In the case of "floating" potential ($F_i = F_e = F$) we obtain by summation of the two equations

$$(20) \quad F\,\frac{k(T_i + T_e)}{e}\,\ln\frac{n_1}{n_2} = \int_\Omega \left[n\alpha \underline{v}_i^2 + n\gamma \underline{v}_e^2 + \beta n(\underline{v}_i - \underline{v}_e)^2 \right] dV$$

n_1, n_2 are the boundary values of n on $\dot{\Omega}_1$ and $\dot{\Omega}_2$.

The ion-neutral collisions and electron-neutral collisions contribute to the losses, and the last term in (20) is the result of Coulomb collisions ($\int \eta\, \underline{j}^2\, dV$).

IV. ELECTRON-NEUTRAL COLLISIONS NEGLECTED

If we neglect the electron-neutral collisions ($\gamma = 0$), the ellipticity of the system is not destroyed. Usually the collision frequencies ν_{io}, ν_{ei} are small compared with the cyclotron frequencies. In this case the velocities can be written:

$$v_{i,\perp} = \frac{\nabla\psi_i \times \underline{B}}{B^2} + (\alpha + \beta)\frac{\nabla_\perp \psi_i}{B^2} + \beta\,\frac{\nabla_\perp \psi_e}{B^2}$$

$$(21) \quad v_{e,\perp} = -\frac{\nabla\psi_e \times \underline{B}}{B^2} + (\gamma + \beta)\frac{\nabla_\perp \psi_e}{B^2} + \beta\,\frac{\nabla_\perp \psi_i}{B^2}$$

$$v_{e,\parallel} = \frac{\beta\,\nabla_\parallel \psi_i + (\alpha + \beta)\nabla_\parallel \psi_e}{\alpha\gamma + \beta(\alpha + \gamma)}\;;\quad v_{i,\parallel} = \frac{(\gamma + \beta)\nabla_\parallel \psi_i + \beta\,\nabla_\parallel \psi_e}{\alpha\gamma + \beta(\alpha + \gamma)}$$

We put $\gamma = 0$ and assume $\alpha \ll \beta$

From the equations of continuity we find

$$\operatorname{div} \frac{2\beta n}{B^2} \nabla_\perp (\psi_i + \psi_e) + \operatorname{div} \frac{2n}{\alpha} \nabla_{\parallel} (\psi_i + \psi_e) + \operatorname{div} \frac{n}{B^2} \nabla(\psi_i - \psi_e) \times \underline{B}$$

(22) $$= Q_i + Q_e$$

$$\operatorname{div} \alpha n \frac{\nabla_\perp \psi_i}{B^2} - \operatorname{div} \frac{n}{\beta} \nabla_{\parallel} \psi_e + \operatorname{div} n \frac{\nabla(\psi_i + \psi_e) \times \underline{B}}{B^2} = 0$$

or, if we go back to the quantities ϕ, n $\quad (Q_i = Q_e = Q)$

$$\psi_i + \psi_e = - \frac{K(T_i + T_e)}{e} \ln \frac{n}{n_0} ; \quad \psi_i - \psi_e = -2\phi + \frac{K(T_e - T_i)}{e} \ln \frac{n}{n_0}$$

With $T_i = T_e$ we find

$$\operatorname{div} \frac{\beta KT}{B^2 e} \nabla_\perp n + \operatorname{div} \frac{KT}{\alpha e} \nabla_{\parallel} n + \operatorname{div} \frac{n}{B} \frac{\nabla\phi \times \underline{B}}{B} = - Q$$

(23)

$$\operatorname{div} \frac{\alpha n}{B^2} \nabla_\perp \phi + \operatorname{div} \frac{n}{\beta} \nabla_{\parallel} \phi + \operatorname{div} \frac{\alpha KT}{e B^2} \nabla_\perp n - \operatorname{div} \frac{KT}{e\beta} \nabla_{\parallel} n + \frac{2KT}{e} \operatorname{div} \frac{\nabla n \times \underline{B}}{B^2} = 0$$

The first equation determines n if ϕ is given, and the second equation determines ϕ if n is known.

Since $\alpha \ll \beta$, $\nabla_{\parallel} n \approx O(\alpha)$ and $\nabla_\perp n \lesssim O(1)$, we may neglect the third and fourth terms in the second equation. All approximations have to be done without changing the ellipticity of the system.

The result is

$$\operatorname{div} \left\{ \frac{\beta KT}{e B^2} \nabla_\perp n + \frac{KT}{\alpha e} \nabla_{\parallel} n + \frac{n}{B^2} \nabla\phi \times \underline{B} \right\} = - Q$$

(24)

$$\operatorname{div} \alpha \frac{n}{B^2} \nabla_\perp \phi + \operatorname{div} \frac{n}{\beta} \nabla_{\parallel} \phi + \frac{2KT}{e} \operatorname{div} \frac{\nabla n \times \underline{B}}{B^2} = 0$$

It is possible to derive an integral relation from eq. (24) which gives the loss flux in terms of the boundary values and a positive functional of (ϕ, n).

The mass velocity is

$$\text{(25)} \qquad \underline{v} = -\frac{\beta KT}{neB^2}\nabla_\perp n - \frac{KT}{n\alpha e}\nabla_\parallel n - \frac{\nabla\phi \times \underline{B}}{B^2}$$

We multiply this equation by ∇n and integrate over the plasma volume:

$$\text{(26)} \qquad \int_\Omega n\underline{v}\cdot\nabla \ln n \, dV = -\int_\Omega \left[\frac{\beta KT}{neB^2}(\nabla_\perp n)^2 + \frac{KT}{\alpha n e}(\nabla_\parallel n)^2\right] dV + \int_\Omega \frac{\nabla n\cdot(\nabla\phi \times \underline{B})}{B^2}\, dV$$

The left-hand side is

$$F \ln n_o - \int Q \ln n \, dV$$

From the second equation of (24) we obtain

$$\text{(27)} \qquad F \ln n_o - \int_\Omega Q \ln n\, dV = -\int_\Omega \left[\frac{\beta KT}{neB^2}(\nabla_\perp n)^2 + \frac{KT}{\alpha n e}(\nabla_\parallel n)^2\right] dV - \int_\Omega \left[\frac{e\alpha n}{2KTB^2}(\nabla_\perp \phi)^2 + \frac{e}{2KT}\frac{n}{\beta}(\nabla_\parallel \phi)^2\right] dV$$

If Q is localized to the magnetic axis and n_1 is the density on the magnetic axis, the left-hand side of eq. (27) becomes

$$- F \ln \frac{n_1}{n_o}$$

F is the loss flux and n_o the density on the boundary.

V. CLOSED-LINE SYSTEMS

We consider a configuration with all magnetic field lines closed. In a certain sense a $l = 2$ stellarator with rational ι ($\iota' = o$) or a multipole without toroidal field are examples of these configurations. In order to describe the magnetic field, we label the field lines by two flux variables ψ, χ. ($\underline{B} = \nabla\psi \times \nabla\chi$). Since $\frac{\alpha}{B} \ll 1$ and $\frac{\beta}{B} \ll 1$, the density and potential are nearly constant along the field lines.

$$n = n_o(\psi, \chi) + o\left(\frac{\alpha}{B}\right)$$

(29)

$$\phi = \phi_o(\psi, \chi) + o\left(\frac{\beta}{B}\right)$$

By expanding with respect to $\frac{\alpha}{B}$ and $\frac{\beta}{B}$ and averaging over the field lines we obtain from eq. (24)

$$< div \frac{\beta KT}{e B^2} \nabla_\perp n_o > + < div \frac{n_o}{B^2} \nabla\phi_o \times \underline{B} > = - < Q >$$

(30)

$$< div \frac{\alpha n_o}{B^2} \nabla_\perp \phi_o > + \frac{2KT}{e} < div \frac{\nabla n_o \times \underline{B}}{B^2} > = 0$$

Here the average is defined by

$$< g > =: \oint g \frac{dl}{B}$$

The last two terms on the left-hand side can be simplified to

$$< div \frac{n_o}{B^2} \nabla\phi_o \times \underline{B} > = \frac{\partial\phi_o}{\partial\psi}\frac{\partial}{\partial\chi} n_o q - \frac{\partial\phi_o}{\partial\chi}\frac{\partial}{\partial\psi} n_o q$$

and

$$=: [\phi_o, n_o q]$$

$$< div \frac{\nabla n_o \times \underline{B}}{B^2} > = [n_o, q]$$

Here

$$q =: \oint \frac{dl}{B}$$

The operators L_1, L_2, which are defined by

$$L_1 n_o = < div \frac{\beta KT}{e B^2} \nabla_\perp n_o > ; \quad L_2 \phi_o =: < div \frac{\alpha n_o}{B^2} \nabla_\perp \phi_o >$$

are second-order hermitian elliptic differential operators in ψ and χ .
The equations are written in the form

$$(32) \qquad \begin{aligned} L_1 n_o + [\phi_o, n_o q] &= - \langle Q \rangle \\ L_2 \phi_o + \frac{2KT}{e} [n_o, q] &= 0 \end{aligned}$$

In a configuraton where the electric field vanishes, the density is constant on the surface q = const $[n_o = n_o(q)]$. But n_o is also determined by $L_1 n_o = \langle Q \rangle$.
The solution of this equation depends on the form of $\langle Q \rangle$ and on the shape of the domain Ω , and, in general, this solution is in contradiction with $n_o = n_o(q)$.

Especially if the boundary (n_o = const) does not coincide with a surface q = const, a solution without electric field is not possible. In the experiment this case occurs if an obstacle is introduced into the plasma; in a closed line system this gives rise to convective motion ($\phi_o \neq 0$) which is described by eq. (28). This situation is similar to the thermal convective motion initiated by inhomogeneities on the boundary (thermal convection on a sunny hill). A localized mass source can also give rise to convection in the plasma. The same arguments as above hold if

$$\langle Q \rangle \neq f(q)$$

In fact, it can be shown that the equations (28) have a strong similarity to the equation of thermal convection of a fluid. We consider a two-dimensional problem and neglect the inertial forces.

$$(33) \qquad \begin{aligned} 0 &= - \nabla p + \rho \nabla U - \alpha \underline{v} \\ - \chi \Delta T + \underline{v} \cdot \nabla T &= \sigma \\ \operatorname{div} \underline{v} = 0 \quad &; \quad v_n = 0 \text{ on } \dot{\Omega} \end{aligned}$$

σ = heat production, χ = heat conductivity, U (x,y) = gravitational potential

$\alpha \underline{v}$ = friction term, $\rho = \rho_o(1- \gamma(T-T_o))$, γ = thermal compressibility.

The term $\alpha \underline{v}$ describes the interaction with a background, and a viscosity term would only slightly change the mathematics of the system. The velocitiy $\underline{v}$ can be described in terms of a stream function $\psi(x,y)$, and by a small calculation we find

$$-\varkappa \Delta T - [\psi, T] = \sigma$$

(34)
$$-\alpha \Delta \psi - [\rho_o \gamma U, T] = 0$$

$[\,,\,]$ = Poisson brackets.

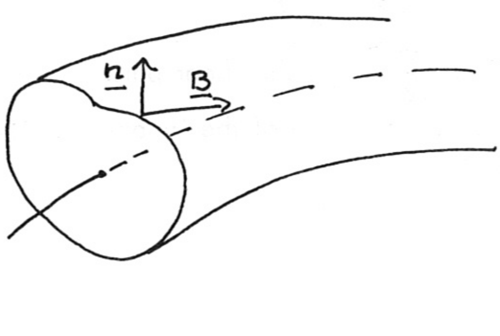
These equations (34) are very similar to (32). The role of the gravitational potential U (x,y) is played by $q(\psi, \chi)$. Furthermore, we identify n_o with T and ϕ_o with ψ.

Summarising, we conclude that in a closed-line system convection is initiated by

1) a localized mass source
2) non-coincidence of the boundary Ω with a surface q = const.

VI. TOROIDAL SYSTEM WITH SHEAR

We consider a magnetic field configuration such as in a stellarator, assuming the existence of magnetic surfaces and the presence of shear. The surfaces are defined by ψ = const ($\underline{B} = \nabla\psi \times \nabla\chi$).



The plasma source may be localized on the magnetic axis and the boundary Ω may coincide with a magnetic surface. We shall solve equations (24) approximately for $\frac{\alpha}{B} = \frac{\nu_{ia}}{\Omega_i} \ll 1$.

From the first equation (24) we find $n = n_o(\psi) + o\left(\frac{\alpha}{B}\right)$.
We insert $n_o(\psi)$ into the second equation and calculate ϕ. The ϕ is introduced into the first equation and this equation is used to find $n_o(\psi)$. Integration over a magnetic surface yields

$$F = \int \frac{\beta KT}{e B^2} \nabla_\perp n_o \cdot \underline{df} + \int \frac{n_o}{B^2}(\nabla\phi \times \underline{B}) \cdot \underline{df}$$

or

(35) $$F = n_o n_o'(\psi) \int \frac{\beta KT}{e B^2} \nabla_\perp n_o \cdot \underline{df} + \int \frac{n_o}{B^2}(\nabla\phi \times \underline{B}) \cdot \underline{df}$$

$\left(\frac{\beta}{n_o}\right.$ is independent of $\left. n_o \right)$.

The potential is calculated from

(36) $$\mathrm{div}\left\{ \frac{\alpha n_o}{B^2} \nabla_\perp \phi + \frac{n_o}{\beta} \nabla_\parallel \phi + \frac{2KT}{e} \frac{\nabla n_o \times \underline{B}}{B^2} \right\} = 0$$

This is an inhomogeneous elliptic equation for ϕ, and this Dirichlet problem always has a solution. We solve this equation approximately by neglecting the first term. This is possible in so far as no "resonant" surface exists on which $\nabla_\parallel \phi$ vanishes. In this case the first term has to be included in the vicinity of this resonant surface. The size of this region is determined by the shear. Some rough estimates give the following results:

(37)
$$\mathrm{div}\, \frac{\alpha n_o}{B^2} \nabla_\perp \phi \approx \frac{\alpha n_o}{B^2} \frac{\phi}{x^2}$$

$$\mathrm{div}\, \frac{n_o}{\beta} \nabla_\parallel \phi \approx \frac{n_o}{\beta} \Theta^2 \frac{x^2}{r^2} \frac{\phi}{\lambda_\perp^2}$$

x is the distance from the resonant surface, $\Theta = \frac{r^2}{R} \iota'$ is the shear parameter, and r = plasma radius, R = torus radius, ι' = shear, $\lambda_\perp$ is a measure of the inhomogeneity of ϕ on the magnetic surface $(\lambda_\perp \ll r)$. $\left(\iota = \frac{\iota}{2\pi}\right)$

The size of the resonant region is defined by

$$div \frac{\alpha n_o}{B^2} \nabla_\perp \phi \approx div \frac{n_o}{\beta} \nabla_{\parallel} \phi$$

By comparing the two terms we find

(38) $$\delta^2 \approx x^2 \leq \frac{r^2}{\Theta} \sqrt{\frac{\nu_{io}}{\Omega_i} \frac{\nu_{ei}}{\Omega_e}}$$

In the neighbourhood of the boundary as wall the first term has to be taken into account. The size of this boundary sheath can be found roughly from

$$\frac{\alpha n_o \phi}{B^2 d^2} \approx \frac{n_o}{\beta} \frac{\phi}{L_{\parallel}^2}$$

$L_{\parallel} < 2\pi R$ is a measure of the parallel variation of ϕ.
The result is

(39) $$d^2 \leq 4\pi^2 R \frac{\nu_{io}}{\Omega_i} \frac{\nu_{ei}}{\Omega_e}$$

The size of this boundary sheath is usually very small. In this case the boundary conditions on ϕ have little effect on the behaviour of ϕ inside Ω.

If the boundary $\dot{\Omega}$ does not coincide with a magnetic surface, the same argumentation holds for all magnetic surfaces which do not touch the boundary. Thus, it follows that in configurations with magnetic surfaces and shear the boundary conditions on ϕ and n as well as the shape of the boundary have little or nearly no effect on the density and potential inside the "last" magnetic surface.

We now consider eq. (36) outside the resonant region.

$$\text{(40)} \qquad \operatorname{div} \frac{n_o}{\beta} \nabla_{\parallel} \phi + \frac{2KT}{e} \nabla n_o \times \underline{B} \cdot \nabla \frac{1}{B^2} = 0$$

In order to solve this equation we introduce a coordinate system ψ, U, V with $\underline{B} = \nabla U = \nabla\psi \times \nabla\chi$ and $(\nabla\psi \times \nabla U)\cdot\nabla V = B^2$. Since the magnetic potential U of a stellarator field is single-valued in the poloidal direction, U plays the role of an azimuthal coordinate. V is the poloidal coordinate and is defined by

$$V = \chi - \iota U$$

Here the rotational transform ι is defined by $\delta\chi - \iota\delta U = 0$, $\delta\chi$ and δU are the periods of χ and U in the azimuthal direction.

In these coordinates equation (40) is written

$$\text{(41)} \qquad \frac{n_o}{\beta}\left(\frac{\partial}{\partial U} - \iota\frac{\partial}{\partial V}\right)^2 \phi + \frac{2KT}{e} n_o'(\psi) \frac{\partial}{\partial V}\left(\frac{1}{B^2}\right) = 0$$

Since $\frac{1}{B^2}$ is periodic in U and V we expand $\frac{1}{B^2}$ in a Fourier series and solve eq. (37) by Fourier expansion.

$$\text{(42)} \qquad \frac{1}{B^2} = \sum g_{m,n} \exp\left[2\pi i \left(m\frac{U}{U_o} + n\frac{V}{V_o}\right)\right]$$

$U_o =: \delta U; V_o = \delta V = \delta \chi$ in the poloidal direction.

With this expansion we find

$$(43) \qquad \frac{n_o}{\beta} 4\pi^2 \left(\frac{m}{U_o} - \iota \frac{n}{V_o}\right)^2 \phi_{m,n} = \frac{2KT}{e} n_o' \frac{2\pi i n}{V_o} g_{m,n}$$

or with $\quad t =: \iota \dfrac{U_o}{V_o}$

$$(44) \qquad \phi_{m,n} = i n_o' \frac{\beta}{n_o} \frac{KT U_o^2}{e\pi V_o} \frac{g_{m,n}(\psi)}{(m - nt)^2}$$

A resonant surface $\psi = \psi_o$ is defined by

$$m - nt(\psi_o) = 0 \quad ; \quad g_{m,n}(\psi_o) \neq 0$$

On this surface $\phi_{m,n}$ diverges and we have to take into account the first term of eq. (36)

Since $\delta \ll r$, we may approximate the operator div $\frac{\alpha n_o}{B^2} \nabla_\perp \phi$ by

$$\mathrm{div} \frac{\alpha n_o}{B^2} \nabla_\perp \phi \approx \frac{\alpha n_o}{B_o^2} \frac{\partial^2 \phi}{\partial x^2}$$

$$n_o \approx n_o(\psi_o), \quad g_{m,n} \approx g_{m,n}(\psi_o) \; ; \; t = t(\psi_o) + t'(x - x_o)$$

B_o is an average value of B on the resonant surface.
The equation (36) becomes

$$(45) \quad -\frac{\alpha n_0}{B_0^4}\frac{\partial^2 \phi_{m,n}}{\partial x^2} + \frac{n_0 4\pi^2}{\beta U_0^2} n^2 t'^2 x^2 \phi_{m,n} = \frac{2KT}{e}\frac{2\pi i}{V_0} n_0' g_{m,n}{}^n$$

with the approximate solution

$$\phi_{m,n} \approx i n_0' \frac{\beta}{n_0}\frac{KT}{e}\frac{U_0^2}{V_0}\frac{n\, g_{m,n}}{n^2 t'^2 x^2 + a}$$

(46)

$$a = n t' \sqrt{\frac{U_0^2}{4\pi B_0^2}\frac{\nu_{io}}{\Omega_i}\frac{\nu_{ei}}{\Omega_e}}$$

Since in a torus with major radius R: $U_o \approx 2\pi B_o R$, a is simplified to

$$a \approx n t' R \sqrt{\frac{\nu_{io}}{\Omega_i}\frac{\nu_{ei}}{\Omega_e}}$$

The characteristic length of this function is

$$\delta^2 \approx \frac{R}{n t'}\sqrt{\frac{\nu_{io}}{\Omega_i}\frac{\nu_{ei}}{\Omega_e}}$$

For n = 1 this is the same formula as given by eq. (38).

For further calculation we will use the following approximation for $\phi_{m,n}$

$$(47) \quad \phi_{m,n} \approx i \frac{n_0'}{n_0}\beta\frac{2KT}{2\pi e}\frac{U_0^2}{V_0}\frac{n\, g_{m,n}(\psi)}{(m-nt)^2 + a}$$

This approximation is good on the singular surface ($m - n\iota = o$) and at a large distance ($x \gg \delta$) from the singular surface. The convective mass flow through a magnetic surface is given by the last term in eq. (35)

$$F_{conv.} = n_o \int \frac{\nabla\phi \times B}{B^2} \cdot df$$

The surface element on the magnetic surface is $df = |\nabla\psi| \frac{dU\, dV}{B^2}$

and

$$(\nabla\phi \times \underline{B})\nabla\psi = B^2 \frac{\partial\phi}{\partial V}$$

The flux is

$$F = n_o \int \frac{\partial\phi}{\partial V} \frac{dU\, dV}{B^2} \tag{48}$$

We insert eq. (47) and (42) in (48) and find after integration over U, V

$$F = n n_o' \beta \frac{2KT}{e} \frac{U_o^2}{V_o^2} \sum_{\substack{m=0\\ n=1}}^{\infty} U_o V_o \frac{n^2 |g_{m,n}|^2}{(m - n\iota)^2 + a} \tag{49}$$

In an axisymmetric configuration (Multipole, Levitron, Spherator) the magnetic field does not depend on the azimuthal coordinate U ($g_{m,n} = o$ for $m \neq o$).

In this case there are no resonant surfaces and the flux is

$$F = n n_o' \beta \frac{2KT}{e} \frac{U_o^2}{V_o^2} U_o V_o \sum_{n=1}^{\infty} \frac{n^2 |g_{o,n}|^2}{n^2 \iota^2 + a} \tag{50}$$

Since $a \ll t$, eq. (50) agrees with the Pfirsch-Schlüter formula. The deviations from axisymmetry introduce the resonance effect. Since the Fourier series $\sum g_{m,n} \exp\left[i 2\pi \left(m \frac{u}{u_o} + n \frac{V}{V_o} \right) \right]$ is convergent, the elements $g_{m,n}$ decrease with $m, n \rightarrow \infty$. Therefore, the "high order resonances" ($t = \frac{m}{n}$, $m, n \gg 1$) do not contribute as much to the loss as the "resonances" at low values of m and n.

If we truncate the Fourier series at some finite values of m and n, the sum given in equation (49) becomes smaller.

Therefore, a calculation of the density profile from eq. (49) with a finite number of Fourier coefficients yields an upper bound of the density. This is also the case if we select a certain class of coefficients of the sum (49) (for example all coefficients with $m - nt = o$) . On a rational surface ($t = \frac{m}{n}$) the following estimate is possible

(51) $$\sum \frac{n^2 |g_{m,n}|^2}{(m-nt)^2 + a} > \sum_{n=1}^{\infty} \frac{n^2 |g_{o,n}|^2}{n^2 t^2 + a} + \sum_{\substack{m=1 \\ n=1}}^{\infty} \frac{n^2 |g_{m,n}|^2}{a}$$

Since $a = n t' R \sqrt{\frac{\nu_{io}}{\Omega_i} \frac{\nu_{ei}}{\Omega_e}} = n a_1$ and $n \geq 1$, a further estimate of the last term in (51) is

(52) $$\sum \frac{n^2 |g_{m,n}|^2}{n a_1} \geq \frac{1}{a_1} \sum_{\substack{m=1 \\ n=1}} |g_{m,n}|^2$$

$$m - nt = 0$$

The sum $\sum (g_{m,n})^2$ can be simplified. From $\chi = V + \iota U$ and eq. (42) we obtain

$$(53) \qquad g_{m,n} = \frac{1}{U_0 V_0} \iint \frac{1}{B^2} e^{-2\pi i \frac{n}{V_0} \chi} \, dU \, dV$$

χ = const along a field line and $dU = Bdl$, $dUdV = d\chi \; Bdl$.
The period in χ is $\chi_0 = \iota U_0 = t \; V_0$.

With $q(\chi) = \oint \frac{dl}{B}$ we obtain

$$(54) \qquad g_{m,n} = \frac{1}{U_0 V_0} \int_0^{\chi_0} q(\chi) e^{-2\pi i m \frac{\chi}{\chi_0}} d\chi$$

According to Parceval's equation the sum over the coefficients $(g_{m,n})^2$ is

$$\sum_{\substack{m=0 \\ m-nt=0}} \frac{(U_0 V_0)^2}{\chi_0^2} |g_{m,n}|^2 = \frac{1}{\chi_0} \int_0^{\chi_0} q^2(\chi) \, d\chi =: \langle q^2 \rangle$$

Together with $\langle q \rangle = \frac{1}{\chi_0} \int_0^{\chi_0} q(\chi) d\chi$ we obtain the result

$$(55) \qquad \sum_{\substack{m=1 \\ m-nt=0}} |g_{m,n}|^2 = \left(\frac{\chi_0}{U_0 V_0} \right)^2 (\langle q^2 \rangle - \langle q \rangle^2) = \frac{t^2}{U_0^2} \langle (q - \langle q \rangle)^2 \rangle$$

If the Newcomb condition q = const on rational surfaces is not satisfied, a convective plasma loss results.

Another estimate for the convective loss flux can be obtained by taking into account only one Fourier coefficient in the second term. $t(\psi)$ may vary between t_1 and t_2 from the magnetic axis to the boundary. We consider all rational values of t between t_1 and t_2 and define $t_0 = \frac{M}{N}$ by

$$M = \min\{m\}\, ;\ N = \min\{n\}\ ;\ t_1 \le \frac{m}{n} \le t_2$$

The following estimate holds

$$\sum \frac{n^2 |g_{m,n}|^2}{(m - nt')^2 + a} \ge \sum_{n=1} \frac{n^2 |g_{o,n}|^2}{n^2 t'^2 + a} + \frac{N^2 |g_{M,N}|^2}{(M - Nt')^2 + a} \qquad (56)$$

With the expansion $t = t_0 + t'\delta\psi$ we find for the last term $\quad (\delta\psi =: \psi - \psi_0)$

$$\frac{N^2 |g_{M,N}|^2}{N^2 t'^2 (\psi - \psi_0)^2 + a}$$

The maximum is given by $\frac{N |g_{M,N}|^2}{a_1}$ and the decay length by $(\delta\psi)^2 = \frac{a}{N^2 t'^2}$

In the approximation of cylindrical geometry the decay length is

$$\delta^2 \approx \frac{rR}{N\delta t} \sqrt{\frac{\nu_{io}}{\Omega_i} \frac{\nu_{ei}}{\Omega_e}} \qquad (57)$$

δt is the change of t over the plasma radius.

The Pfirsch-Schlüter term is roughly

$$F_{P.S.} \sim \frac{|g_{o1}|^2}{t'^2} \qquad (58)$$

On the rational surface $\iota_0 = \frac{M}{N}$ the relation between convective loss and Pfirsch-Schlüter loss is

$$\frac{F_{conv}}{F_{P.S.}} \approx \frac{\iota^2 N}{\delta_t} \frac{r}{R} \frac{|g_{M,N}(\psi_0)|^2}{\sqrt{\frac{\nu_{io}}{\Omega_i}\frac{\nu_{ei}}{\Omega_e}}\,|g_{0,1}|^2} \qquad (59)$$

The density profile can be obtained by integrating eq. 49. It depends on the detailed shape of the coefficients $g_{m,n}(\psi), \iota(\psi)$, and $\iota'(\psi)$. To illustrate the dependence of the density on ι we choose an example with $g_{m,n}(\psi) = \text{const} = A$ $\iota \approx \text{const}, \iota' = \text{const}$. In a certain respect this example is verified by l = 2 stellarator. The density on the magnetic axis is shown in fig. 3.

The answer for an l=3 stellarator is more difficult to give. One has to distinguish between two cases. If there is only one resonant region, the shear localizes the convection zone to the neighbourhood of this resonant region.

If there are more than one resonant region, then the convection zones can overlap and the whole plasma region is affected by convective losses.

As we see from eq. (42) in axisymmetric configurations, these resonant regions cannot occur since

$$g_{m,n} = 0 \quad \text{for} \quad m = 1,2,\dots$$

But small deviations from axisymmetry due to imperfections of the coil system or the effect of current leads can introduce resonant regions into the magnetic field. In stellarators the helical field yields a deviation from axisymmetry. Whether or not a resonant region occurs depends on the detailed behaviour of $\oint \frac{dl}{B}$ on rational magnetic surfaces.

In the last few years this problem has been tackled many times, but a general property of $\oint \frac{dl}{B}$ in toroidal vacuum fields could not be found.

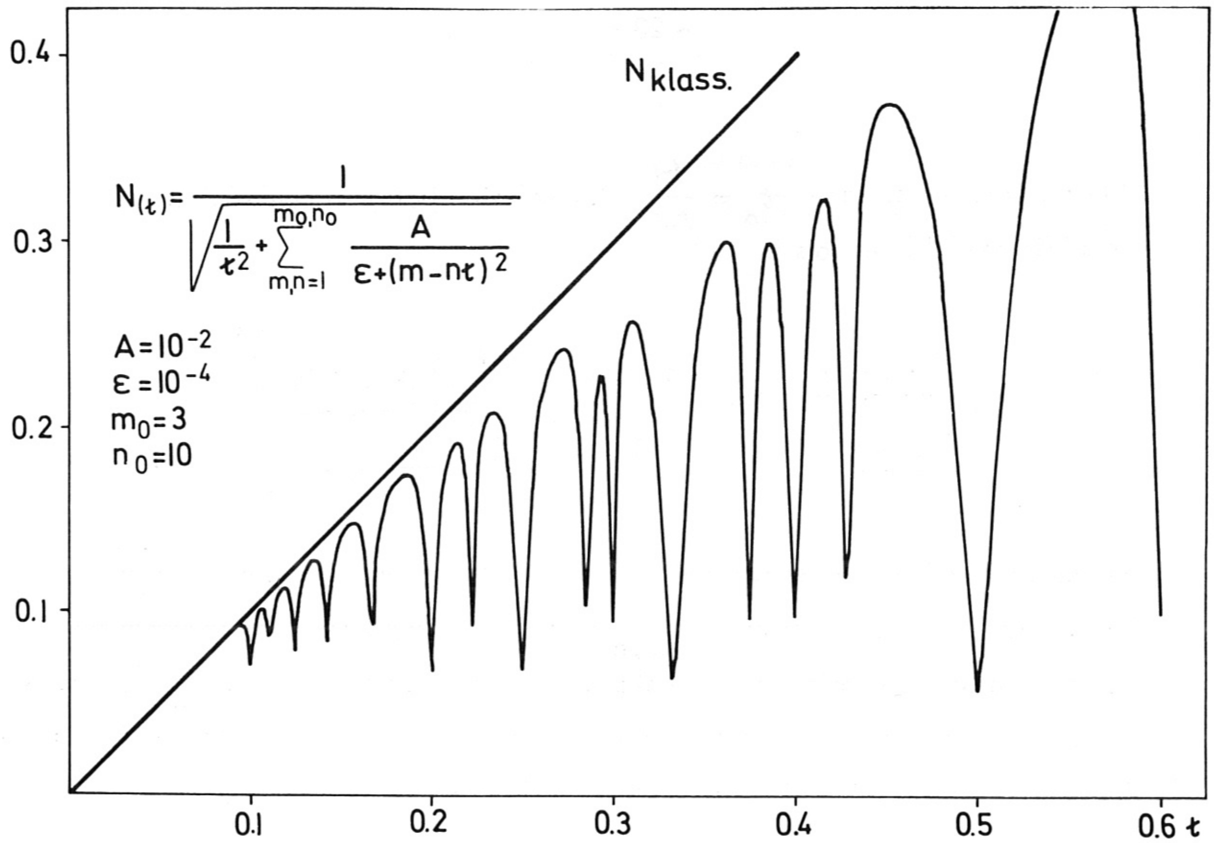


Fig. 3

VII. CONCLUSIONS

It has been shown that by including of plasma-neutral interaction, a quasilinear elliptic system for the plasma density and electric potential in stationary equilibrium can be derived. Together with Dirichlet boundary conditions we therefore obtain a well posed problem. In general, this nonlinear system has more than one solution, and so the interesting question of bifurcation points arises.

The equations for n and $\emptyset$ have the same mathematical form as the equations for thermal convection. In a closed-line system the solution has a very sensitive dependence on the boundary conditions and on the mass source. In a magnetic field with shear the convective solution is localized to a "resonant" surface ($M - N t = o$). In this case the boundary conditions are not so important, but here the structure of the magnetic field (asymmetries) determine the convective solution. In the case of low shear and "resonant" values of t it has been shown that small asymmetries can lead to a large convective loss.

A still unsolved problem is the stability of this convective solution. Here we expect similarity with the case of thermal convection. The variation of the density in the magnetic surface should also be included. This was done by S. Yoshikawa [6] in a slab geometry, where he found "pseudoclassical" diffusion.

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